METRIC (IN) DEPENDENCE OF THE PARTITION FUNCTION OF CHERN-SIMONS THEORY

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One of the motivations for the recent interest on the study of Chern-Simons theories is based on its rich topological structure, which opens new possibilities for the definition of topological invariants of knots, links and three-dimensional manifolds. Because of the general covariant properties of the classical theory it is natural to expect that upon quantization it would provide quantum field theoretical definitions for those invariants. This beautiful application was first remarked by Schwarz in the case of abelian Chern-Simons theories [1] and more recently generalized by Witten for non-abelian Chern-Simons theory [2].

However from a field theoretical viewpoint it is always tantalizing to see whether the process of quantization preserves all the classical symmetries and there are not gravitational or topological anomalies. In the canonical formalism it turns out to be case [3]. However, the study of possible gravitational or framing anomalies is not complete in the canonical formalism because of the special form of space-metric in such a formalism: direct product of a two-dimensional Riemannian metric and an one-dimensional time scale, e.g. the induced gravitational Chern-Simons term is not metric dependent for such metrics, and vanish for some choices of framing. Therefore, the study of this problem for arbitrary three-dimensional manifolds requires a deeper understanding of the quantization of field theories on arbitrary space-time manifolds. Although, the analysis of the anomalies can be achieved in an indirect way by using topological techniques like topological surgery, a complete discussion in terms of pure field theoretical arguments requires the use of a three-dimensional covariant approach.

The theory is finite and solvable in the canonical formalism for compact Lie groups in absence of space-time boundaries and external sources (spatial punctures, local matter, etc.), but in the covariant formalism it is simply renormalizable. The existence of singularities is due to the presence of some unphysical degrees of freedom in this formalism. Therefore, the analysis becomes non trivial and to some extent the fluctuations of those unphysical degrees of freedom might veil the topological nature of the theory.

The standard mathematical methods used in the analysis of quantum anomalies by means of index theorems (ζ-function and heat kernel) only hold for one-loop generated anomalies. Since in this case gravitational anomalies might also be generated by higher order corrections it becomes necessary to consider general methods of quantum field theory. Because of the pseudoscalar character of the Chern-Simons action, standard perturbative regularization methods can not be applied. This fact, has recently stimulated the interest on the application of different perturbative regularization prescriptions to Chern-Simons theories [4]-[6],[8].

We shall use a geometrical regularization which preserves many of the interesting properties of the model: continuity of space-time and invariance under framing and gauge transformations. The regularization is based on the observation that, because of gauge invariance, the relevant space of covariant field configurations is also the space of covariant gauge orbits $\mathcal{M}$. Since $\mathcal{M}$ is a curved ∞-dimensional Riemannian manifold the regularization of a functional integral defined over $\mathcal{M}$ does not simply requires a regularization of the action, as in ordinary field theories with flat configuration spaces, but also a non-trivial regularization of the functional (Riemannian) volume element [9]. In this way it is possible to obtain a regularization which overcomes the problem of overlapping divergences usually associated to the regularization by means of higher covariant derivatives and Pauli-Villars ghosts, preserves the topological properties of continuum approaches and has a non-perturbative interpretation (see Refs. [5][7]).

In the Landau gauge $d^*_{A_0}(A_c - A_0) = 0$ the geometrically regularized partition function reads

$$\int \delta A_c \; det_c^{1/2} g^0_{\Lambda'}(A_c) \; det_c^{1/2} (G^1_{\Lambda'})^{-1} \; det_c^{1/2} (G^2_{\Lambda'})^{-1} G^1_{\Lambda'} \; det^{1/2}_{\Lambda\Lambda'} \Delta^0_{A_c} \, \mathrm{e}^{-S_\Lambda(A_c)}, \tag{1}$$

where $g^0_{\Lambda'}$ represents a strong, covariant metric of Sobolev type on the orbit space $\mathcal{M}$, G^i_Λ are trace class operators with respect to the metric $g^0_{\Lambda'}$, the gauge invariant factor $det^{1/2}_{\Lambda\Lambda'}\Delta^0_A$ stands for the regularized volume of the gauge orbits and, finally, S_Λ is the action of Chern-Simons regularized by the addition of a Yang-Mills term with higher covariant derivatives.

In flat space-time, for appropriate choices of the regulating operators, the regularization is consistent and preserves gauge invariance. One loop fluctuations generate in flat space-times a finite renormalization [5] of the Chern-Simons coupling $k \to k + N$ (for $SU(N)$), which is expected to hold in higher orders in perturbation theory [10].

However to analyse the existence of diffeomorphisms anomalies we must carry out the quantization of the theory in arbitrary metric backgrounds and look at the possible metric dependence of the resulting theory. A metric dependence of the Wilson loop expectation values has been detected in Refs. [11]. However, the physical observable which is most sensitive to a change of space-time metric is the partition function $\mathcal{Z}$. This is due to the fact that $\mathcal{Z}$ contains the leading cubic divergences, which also generate the loss of framing independence associated to the non-trivial transformation of two-dimensional partition function under conformal transformations in the two-dimensional conformal field theories.

Although the classical lagrangian is metric independent a metric dependence might appear in the partition function from the functional measure δA_c [12] or the gauge fixing condition because both are metric dependent. Moreover, since geometric regularization is defined by means of a binuclear Riemannian structure of the orbit space which depends on the space-time metric, the metric dependence generated by the regularization could remain even after the removal the ultraviolet regulators ($\Lambda, \Lambda' \to \infty$).

The one loop contribution given by

$$\mathcal{Z}^{(1)}(g) = \frac{\det_c^{1/2}(I + \Delta^1/\Lambda'^2)^{n+1} \ \det^{1/2}\left(\Delta^0/\Lambda^2(I + \Delta^0/\Lambda^2)^{2n} + 1\right) \ \det^{1/2}\Delta^0}{\det_c^{1/2}\left[\Delta^1/\Lambda^2(I + \Delta^1/\Lambda^2)^n + i * d\right] \ \det^{1/2}(I + \Delta^0/\Lambda'^2)^{2n+2}} \tag{2}$$

in the generalized Landau gauge ($d^*_{A_0}(A - A_0) = 0$) is finite and can be evaluated in the weak limit approximation $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. The result is of the form [13]

$$\mathcal{Z}^{(1)} = \mathcal{Z}_0^{(1)}(g) \mathrm{e}^{c_1 \int_{M_3} \sqrt{g} + c_2 \int_{M_3} \sqrt{g} R + \frac{ic_3}{4\pi} \int_{M_3} \mathrm{Tr}(\omega\wedge\omega + 2/3\omega\wedge\omega\wedge\omega) + \mathcal{O}(1/\Lambda)} \tag{3}$$

where $\mathcal{Z}_0^{(1)}(g)$ contains the non-local terms induced by the global framing anomaly [2] and non-perturbative contributions generated by the existence of zero modes [13]. We remark that once the Green functions are finite in flat space-times they remain finite for arbitrary space-time metrics. Therefore there are no further restrictions on the parameters of geometric regularization.

The only terms which remain finite in the ultraviolet limit, $\Lambda, \Lambda' \to \infty$, are the gravitational Chern-Simons term c_3 and $\mathcal{Z}_0^{(1)}(g)$. The other terms would require counterterms to cancel their divergent contributions, $c_1 = \mathcal{O}(\Lambda^3) + \mathcal{O}({\Lambda'}^3)$ and $c_2 = \mathcal{O}(\Lambda) + \mathcal{O}(\Lambda')$.

The explicit calculation of the coefficients of the induced gravitational action (3) yields the following values

$$c_1 = 0, \qquad c_2 = \frac{N^2-1}{16}(\alpha\Lambda - \pi(n+1)\Lambda'), \qquad c_3 = \frac{N^2-1}{24} \tag{4}$$

for $SU(N)$ in the generalized Landau gauge, with

$$\alpha = \int_0^\infty dp \ \frac{1 + 2n(1 + p^2(1+p^2)^{2n-1})}{1 + p^2(1+p^2)^{2n}}. \tag{5}$$

The coefficient c_2 of the Einstein-Hilbert term depends on the parameters of the regularization and can be cancelled by a suitable choice of the regulators masses, $\Lambda' = \alpha\Lambda/\pi(n+1)$. However, the coefficient of the Chern-Simons term is universal and cannot be cancelled by any choice of the parameters of the regularization.

The value of $c_3 = (N^2 - 1)/24$ is in agreement with the exact value conjectured by Witten

$$c_3 = \frac{k(N^2-1)}{24(k+N)} = \frac{N^2-1}{24}(1 - \frac{N}{k} + \mathcal{O}(\frac{N^2}{k^2})). \tag{6}$$

A two loop calculation of c_3 has been recently carried out in a different perturbative regularization scheme [8]. The result gives the second term of Witten's expansion (6), but in such a scheme the first term is missing.

In the geometric regularization scheme there is no framing anomaly because the change of the gravitational Chern-Simons term under non-trivial framing transformations is compensated by the change of the non-local part of $\mathcal{Z}_0^{(1)}(g)$. In fact, geometric regularization preserves framing independence explicitly. But, consequently, the partition function becomes metric dependent. The cancellation of this dependence can only be achieved by the addition of a finite gravitational Chern-Simons counterterm which induces a framing anomalous behavior of the quantum partition function. In this sense metric dependence can be traded by a framing topological anomaly.

The one loop contributions to the partition function $\mathcal{Z}^{(1)}(g)$ can also be exactly calculated beyond the weak field expansion for some particular space-time backgrounds $(S^3, \Sigma \times S^1)$ [13]. The results confirm the values of the coefficients (6) of the local terms of the induced action: cosmological, Einstein-Hilbert and gravitational Chern-Simons terms, and provide explicit expressions for the non-local terms of $\mathcal{Z}_0^{(1)}(g)$. In the case of a manifold of the form $\Sigma \times S^1$ the metric dependence of those non-local terms cancels out with that of the non-perturbative contributions of $\mathcal{Z}_0^{(1)}(g)$, up to a constant factor $(\text{vol } M_3)^{1/2}$ [1] which is genus independent [13]. This fact stresses the topological character (metric independent) of Chern-Simons theory in the canonical formalism. Because of the direct product structure of the space-time metric in this formalism the gravitational Chern-Simons term is not metric dependent and there always exist a framing where it vanishes. Once the metric dependent factor is eliminated the partition function of the abelian theory $\mathcal{Z}^{(1)}(g) \sim k^{g+1}$ gives account of vacuum degeneracy k^g [13]. In the non-abelian case it gives only the leading approximation to the exact vacuum degeneracy [13]. It will be very interesting to know whether higher order perturbative calculations agree with higher order corrections to the exact formula.

Although there is not any apparent symmetry argument behind the choice of the masses of the regulators $\Lambda' = \alpha\Lambda/\pi\ (n\ +\ 1\ )$, it is a necessary condition to cancel an explicit metric dependence which arises from the quantum fluctuations. The partition function is not the only observable which picks up quantum metric dependent contributions [11]. In general, it is always possible to factorize the metric dependent contributions and obtain a topological invariant, but this procedure has to be carried out very carefully because some relevant topological information can also be lost by a crude renormalization. This necessity of renormalization introduces some ambiguities in the definition of topological invariants in Chern-Simons theory.

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